ON ALMOST CONVERGENCE FOR VECTOR-VALUED FUNCTIONS AND ITS APPLICATION (Nonlinear Analysis and Convex Analysis)

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1. INTRODUCTION

In 1948, Lorentz [11] introduced a notion of almost convergence for bounded sequences of real numbers: Let $\{x_n\}$ be a bounded sequence of real numbers. Then, $\{x_n\}$ is said to be almost convergent if

$$\mu_n(x_n) = \nu_n(x_n)$$

for any Banach limits μ and ν. Day [6] defined a notion of almost convergence for bounded real-valued functions defined on an amenable semigroup.

On the other hand, von Neumann [15] introduced a notion of almost periodicity for bounded real-valued functions defined on a group and proved the existence of the mean values for those functions. Later, Bochner and von Neumann [3] proved the existence of the mean values for vector-valued almost periodic functions defined on a group with values in a locally convex space. Recently, Miyake and Takahashi [13, 14] proved the existence of the mean values for vector-valued almost periodic functions defined on an amenable semigroup and obtained non-linear mean ergodic theorems for transformation semigroups of various types.

In this paper, we announce some results recently obtained in studying on almost convergence for vector-valued functions defined on an amenable semigroup with values in a locally convex space. First, motivated by the work of Lorentz, we introduce a notion of almost convergence for those functions and obtain characterizations of vector-valued almost convergent functions. Next, we introduce a notion of the mean values for those functions defined on a semigroup without assumption of amenability and prove characterizations of the space of bounded real-valued functions defined on a semigroup. Finally, by study on almost convergence for commutative semigroups of non-linear mappings, we prove mean ergodic theorems for non-Lipschitzian asymptotically isometric semigroups of continuous self-mappings of a compact convex subset of a general Banach space.

2. Preliminaries

Throughout this paper, we denote by S a semigroup with identity and by E a locally convex topological vector space (or l.c.s.). We also denote by $\mathbb{R}_+$ and $\mathbb{N}_+$ the set of non-negative real numbers and the set of non-negative integers, respectively. Let $\langle E, F\rangle$ be the duality between vector spaces E and F. For each $y \in F$, we define a linear functional f_y on E by $f_y(x) = \langle x, y\rangle$. We denote by $\sigma(E, F)$ the weak topology on E generated by $\{f_y : y \in F\}$. E_σ denotes a l.c.s. E with the weak topology $\sigma(E, E')$. If X is a l.c.s., we denote by X' the topological dual of X. We also denote by $\langle\cdot,\cdot\rangle$ the canonical bilinear form between E and E', that is, for $x \in E$ and $x' \in E'$, $\langle x, x'\rangle$ is the value of x' at x.

We denote by $l^\infty(S)$ the Banach space of bounded real-valued functions on S. For each $s \in S$, we define operators $l(s)$ and $r(s)$ on $l^\infty(S)$ by

$$(l(s)f)(t) = f(st) \quad \text{and} \quad (r(s)f)(t) = f(ts)$$

for each $t \in S$ and $f \in l^\infty(S)$, respectively. A subspace X of $l^\infty(S)$ is said to be *translation invariant* if $l(s)X \subset X$ and $r(s)X \subset X$ for each $s \in S$. Let X be a subspace of $l^\infty(S)$ which contains constants. A linear functional μ on X is said to be a *mean* on X if $\|\mu\| = \mu(e) = 1$, where $e(s) = 1$ for each $s \in S$. We often write $\mu_s f(s)$ instead of $\mu(f)$ for each $f \in X$. For $s \in S$, we define a *point evaluation* δ_s by $\delta_s(f) = f(s)$ for each $f \in X$. A convex combination of point evaluations is called a *finite mean* on S. As is well known, μ is a mean on X if and only if

$$\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s)$$

for each $f \in X$; see Day [6] and Takahashi [22] for more details. Let X be also translation invariant. Then, a mean μ on X is said to be *left (or right) invariant* if $\mu(l(s)f) = \mu(f)$ (or $\mu(r(s)f) = \mu(f)$) for each $s \in S$ and $f \in X$. A mean μ on X is said to be *invariant* if μ is both left and right invariant. If there exists a left (or right) invariant mean on X, then X is said to be *left (or right) amenable.* If X is also left and right amenable, then X is said to be *amenable.* We know from Day [6] that if S is commutative, then X is amenable. Let $\{\mu_\alpha\}$ be a net of means on X. Then $\{\mu_\alpha\}$ is said to be *asymptotically invariant (or strongly regular)* if for each $s \in S$, both $l(s)'\mu_\alpha - \mu_\alpha$ and $r(s)'\mu_\alpha - \mu_\alpha$ converge to 0 in the weak topology $\sigma(X', X)$ (or the norm topology), where $l(s)'$ and $r(s)'$ are the adjoint operators of $l(s)$ and $r(s)$, respectively. Such nets were first studied by Day [6].

We denote by $l^\infty(S, E)$ the vector space of vector-valued functions defined on S with values in E such that for each $f \in l^\infty(S, E)$, $f(S) =$

$\{f(s) : s \in S\}$ is bounded. Let $\mathfrak{U}$ is a neighborhood base of 0 in E and let $M(V) = \{f \in l^\infty(S,E) : f(S) \subset V\}$ for each $V \in \mathfrak{U}$. A family $\mathfrak{B} = \{M(V) : V \in \mathfrak{U}\}$ is a filter base in $l^\infty(S,E)$. Then, $l^\infty(S,E)$ is a l.c.s. with the topology $\mathfrak{T}$ of uniform convergence on S that has a neighborhood base $\mathfrak{B}$ of 0. For each $s \in S$, we define the operators $R(s)$ and $L(s)$ on $l^\infty(S,E)$ by

$$(R(s)f)(t) = f(ts) \quad \text{and} \quad (L(s)f)(t) = f(st)$$

for each $t \in S$ and $f \in l^\infty(S,E)$, respectively. Let $f \in l^\infty(S,E)$. We denote by $\mathcal{RO}(f)$ the right orbit of f, that is, the set $\{R(s)f \in l^\infty(S,E) : s \in S\}$ of right translates of f. Similarly, we also denote by $\mathcal{LO}(f)$ the left orbit of f, that is, the set $\{L(s)f \in l^\infty(S,E) : s \in S\}$ of left translates of f. A subspace Ξ of $l^\infty(S,E)$ is said to be *translation invariant* if $L(s)\Xi \subset \Xi$ and $R(s)\Xi \subset \Xi$ for each $s \in S$. Let Ξ be a subspace of $l^\infty(S,E)$ which contains constant functions. For each $s \in S$, we define a *(vector-valued) point evaluation* Δ_s by $\Delta_s(f) = f(s)$ for each $f \in l^\infty(S,E)$. A convex combination of vector-valued point evaluations is said to be a *(vector-valued) finite mean.* A mapping M of Ξ into E is called a *vector-valued mean* on Ξ if M is contained in the closure of convex hull of $\{\Delta_s : s \in S\}$ in the product space $(E_\sigma)^\Xi$. Then, a vector-valued mean M on Ξ is a linear continuous mapping of Ξ into E such that (i) $Mp = p$ for each constant function p in Ξ, and (ii) $M(f)$ is contained in the closure of convex hull of $f(S)$ for each $f \in \Xi$. We denote by Φ_Ξ the set of vector-valued means on Ξ. Let Ξ be also translation invariant. Then, a vector-valued mean M on Ξ is said to be *left (or right) invariant* if $M(L(s)f) = M(f)$ (or $M(R(s)f) = M(f)$) for each $s \in S$ and $f \in \Xi$. A vector-valued mean M on Ξ is said to be *invariant* if M is both left and right invariant. Let $f \in \Xi$ and let M be a vector-valued mean on Ξ. We define a vector-valued function $M.f \in l^\infty(S,E)$ by $(M.f)(s) = M(L(s)f)$ for each $s \in S$. Then, Ξ is said to be *introverted* if for each $f \in \Xi$ and vector-valued mean M on Ξ, $M.f$ is contained in Ξ.

We also denote by $l_c^\infty(S,E)$ the subspace of $l^\infty(S,E)$ such that for each $f \in l_c^\infty(S,E)$, $f(S)$ is relatively weakly compact in E. Let X be a subspace of $l^\infty(S)$ containing constants such that for each $f \in l_c^\infty(S,E)$ and $x' \in E'$, a function $s \mapsto \langle f(s), x'\rangle$ is contained in X. Such an X is called *admissible.* Let $\mu \in X'$. Then, for each $f \in l_c^\infty(S,E)$, we define a linear functional $\tau(\mu)f$ on E' by

$$\tau(\mu)f : x' \mapsto \mu\langle f(\cdot), x'\rangle.$$

It follows from the bipolar theorem that $\tau(\mu)f$ is contained in E. A mapping τ of X' onto $\Phi_{l_c^\infty(S,E)}$ is linear and continuous where X' is equipped with the weak topology $\sigma(X',X)$. Then, for each mean μ on X, $\tau(\mu)$ is a vector-valued mean on $l_c^\infty(S,E)$ (generated by μ). Conversely, every vector-valued mean on $l_c^\infty(S,E)$ is also a vector-valued mean in the sense of Goldberg and Irwin [8], that is, for each $M \in \Phi_{l_c^\infty(S,E)}$, there exists a mean μ on X such that $\tau(\mu) = M$. Note that $\Phi_{l_c^\infty(S,E)}$ is compact and convex in $(E_\sigma)^{l_c^\infty(S,E)}$; see also Day [6], Takahashi [20, 22] and Kada and Takahashi [10]. Let X be also translation invariant and amenable. If μ is a left (or right) invariant mean on X, then $\tau(\mu)$ is also left (or right) invariant. Conversely, if M is a left (or right) invariant vector-valued mean on $l_c^\infty(S,E)$, then there exists a left (or right) invariant mean μ on X such that $\tau(\mu) = M$.

Let C be a closed convex subset of a l.c.s. E and let $\mathfrak{F}$ be the semigroup of continuous self-mappings of C under operator multiplication. If T is a semigroup homomorphism of S into $\mathfrak{F}$, then T is said to be a *representation* of S as continuous self-mappings of C. Let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as continuous self-mappings of C such that for each $x \in C$, the orbit $\mathcal{O}(x) = \{T(s)x : s \in S\}$ of x under $\mathcal{S}$ is relatively weakly compact in C and let X be a subspace of $l^\infty(S)$ containing constants such that for each $x \in C$ and $x' \in E'$, a function $s \mapsto \langle T(s)x, x'\rangle$ is contained in X. Such an X is called *admissible* with respect to $\mathcal{S}$. If no confusion will occur, then X is simply called admissible. Let $\mu \in X'$. Then, there exists a unique point x_0 of E such that $\mu\langle T(\cdot)x, x'\rangle = \langle x_0, x'\rangle$ for each $x' \in E'$. We denote such a point x_0 by $T(\mu)x$. Note that if μ is a mean on X, then for each $x \in C$, $T(\mu)x$ is contained in the closure of convex hull of the orbit $\mathcal{O}(x)$ of x under $\mathcal{S}$.

3. On almost convergence for vector-valued functions

Motivated by the work of Lorentz [11], we introduce a notion of almost convergence for vector-valued functions defined on a left amenable semigroup with values in a locally convex space and also obtain characterizations of almost convergence for those functions.

Definition 1. Let S be left amenable and let $f \in l_c^\infty(S,E)$. Then, f is said to be *almost convergent* in the sense of Lorentz if

$$\tau(\mu)f = \tau(\nu)f$$

for any left invariant means μ and ν on $l^\infty(S)$. Note that f is almost convergent in the sense of Lorentz if and only if $M(f) = N(f)$ for any left invariant vector-valued means on M and N on $l_c^\infty(S,E)$.

Theorem 1. *Let S be left amenable and let $f \in l_c^\infty(S,E)$. Then, the following are equivalent:*

(i) *f is almost convergent in the sense of Lorentz;*

(ii) *the closure $\mathcal{K}$ of convex hull of $\mathcal{RO}(f)$ contains exactly one constant function in the topology τ_{wp} of weakly pointwise convergence on S;*
(iii) *for each function $g \in \mathcal{K}$, the τ_{wp}-closure of convex hull of $\mathcal{RO}(g)$ contains exactly one constant function.*

Theorem 2. *Let S be commutative, let $f \in l_c^\infty(S, E)$ and let X be a closed, translation invariant and admissible subspace of $l^\infty(S)$ containing constant functions. Then, the following are equivalent:*
(i) *f is almost convergent in the sense of Lorentz;*
(ii) *there exists a strongly regular net $\{\lambda_\alpha\}$ of finite means such that $\{\tau(\lambda_\alpha).f\}$ converges in the topology τ_{wu} of weakly uniform convergence on S;*
(iii) *for each strongly regular net $\{\mu_\alpha\}$ of means on X, $\{\tau(\mu_\alpha).f\}$ converges in the topology τ_{wu}.*

Next, we introduce a notion of the mean value for bounded vector-valued functions defined on a semigroup without assumption of amenability and also obtain characterizations of the space of bounded real-valued functions defined on a semigroup which have the mean values.

Definition 2. Let $f \in l^\infty(S, E)$ and let $\mathcal{K}$ be the closure of convex hull of $\mathcal{RO}(f)$ in the topology τ_{wp} of weakly pointwise convergence on S. If for each function g in $\mathcal{K}$, the τ_{wp}-closure of convex hull of $\mathcal{RO}(g)$ contains exactly one constant function with value p, then p is said to be *the mean value* of f; see also von Neumann [15], Bochner and von Neumann [3] and Miyake and Takahashi [13]. In particular, if S is commutative, then it follows from Theorem 1 that $f \in l_c^\infty(S, E)$ has the mean value if and only if the τ_{wp}-closure of convex hull of $\mathcal{RO}(f)$ contains exactly one constant function. We denote by $AC(S)$ the set of bounded real-valued functions defined on S with the mean values.

As in similar arguments of Lemma 1 (the localization theorem) in [9], we obtain some characterizations of the space of bounded real-valued functions defined on a semigroup with the mean values.

Proposition 1. *$AC(S)$ is a translation invariant and introverted subspace of $l^\infty(S)$ containing constant functions.*

Note that it follows from Theorem 1 that if S is left amenable, then $AC(S)$ is the subspace of $l^\infty(S)$ consisting of bounded real-valued functions defined on S which are almost convergent in the sense of Lorentz.

Theorem 3. *$AC(S)$ is amenable and has a unique invariant mean μ. In this case, μ is also a unique left invariant mean on $AC(S)$.*

Theorem 4. *$AC(S)$ is a maximum translation invariant and introverted subspace of $l^\infty(S)$ containing constant functions which has a unique left invariant mean, ordered by set inclusion.*

Theorem 5. *If S is commutative, then $AC(S)$ is a maximum translation invariant subspace of $l^\infty(S)$ containing constant functions which has a unique invariant mean, ordered by set inclusion.*

4. Applications

By studying on almost convergence in the sense of Lorentz for commutative semigroups of non-linear mappings, we prove mean ergodic theorems for non-Lipschitzian asymptotically isometric semigroups of continuous mappings in general Banach spaces. The following lemma is crucial for proving our results.

Lemma 1. *Let S be commutative and let $f \in l_c^\infty(S, E)$. If the closure of convex hull of $\mathcal{RO}(f)$ contains a constant function with value p in the topology of uniform convergence on S, then f is almost convergent in the sense of Lorentz (equivalently, f has the mean value p.)*

Definition 3. Let S be commutative and let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as continuous mappings of a closed convex subset C of a Banach space E into itself. Then, $\mathcal{S}$ is said to be *asymptotically isometric* on C if, for each $x \in C$,

$$\lim_{s \in S} \|T(s+k)x - T(s+h)x\| \quad \text{exists uniformly in } k, h \in S.$$

See Bruck [4] and Kada and Takahashi [10].

Definition 4. Let S be left amenable and let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as continuous mappings of a weakly compact convex subset C of E into itself and define a mapping $\phi_{\mathcal{S}}$ of C into $l_c^\infty(S, E)$ by $(\phi_{\mathcal{S}}(x))(s) = T(s)x$ for each $s \in S$. Then, a representation $\mathcal{S}$ is said to be *almost convergent* in the sense of Lorentz if, for each $x \in C$, $\phi_{\mathcal{S}}(x)$ has the mean value p_x. Such a point p_x is also said to be *the mean value* of x under $\mathcal{S}$.

Theorem 6. *Let S be commutative, let C be a compact convex subset of a Banach space E, let $\mathcal{S} = \{T(s) : s \in S\}$ be an asymptotically isometric representation of S as continuous mappings of C into itself, let X be a closed, translation invariant and admissible subspace of $l^\infty(S)$ containing constants and let $\{\mu_\alpha\}$ be a strongly regular net of means on X. Then, $\mathcal{S}$ is almost convergent in the sense of Lorentz, that is, for each $x \in C$, $\{T(l(h)'\mu_\alpha)x\}$ converges to the mean value p_x of x under $\mathcal{S}$ in C uniformly in $h \in S$. In this case, $p = T(\mu)x$ for each invariant mean μ on X.*

Remark 1. Note that the mean value $T(\mu)x$ of x under $\mathcal{S}$ is not always a common fixed point for $\mathcal{S}$. It is known in [19] that there exists a nonexpansive mapping T of C into itself such that for some $x \in C$, its Cesàro means $\{1/n \sum_{k=0}^{n-1} T^k x\}$ converge, but its limit point is not a fixed point of T; see also Edelstein [7], Bruck [5], Atsushiba and Takahashi [1], Atsushiba, Lau and Takahashi [2], Miyake and Takahashi [13] and Miyake and Takahashi [14]. We conjecture in Theorem 6 that if a Banach space E is strictly convex, then the mean value p_x of x under $\mathcal{S}$ is a common fixed point for $\mathcal{S}$, that is, $T(s)p_x = p_x$ for each $s \in S$.

For example, the following corollaries are the case when S is a set of the non-negative integers or real numbers.

Corollary 1. *Let C be a compact convex subset of a Banach space, let T be a continuous mapping of C into itself such that $\lim_{n\to\infty} \|T^{n+k}x - T^{n+h}x\|$ exists uniformly in $k, h \in \mathbb{N}_+$. Then, for each $x \in C$, the Cesàro means*

$$\frac{1}{n}\sum_{i=0}^{n-1} T^{i+h}x$$

converge to the mean value of x under T in C uniformly in $h \in \mathbb{N}_+$.

Corollary 2. *Let C be a compact convex subset of a Banach space and let $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$ be an asymptotically isometric one-parameter semigroup of continuous mappings of C into itself. Then, for each $x \in C$, the Bohr means*

$$\frac{1}{t}\int_0^t T(t+h)x \, dt$$

converge to the mean value of x under $\mathcal{S}$ in C uniformly in $h \in \mathbb{R}_+$ as $t \to +\infty$.

Corollary 3. *Let C be a compact convex subset of a Banach space and let $\mathcal{S} = \{T(t) : t \in \mathbb{R}_+\}$ be an asymptotically isometric one-parameter semigroup of continuous mappings of C into itself. Then, for each $x \in C$, the Abel means*

$$r\int_0^\infty \exp(-rt)T(t+h)x \, dt$$

converge to the mean value of x under $\mathcal{S}$ in C uniformly in $h \in \mathbb{R}_+$ as $r \to +\infty$.

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